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Colliding Plane Gravito-Electromagnetic Waves

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The collision and interaction of plane electromagnetic waves is considered. Field equations are formulated and the appropriate boundary conditions are described in detail. A solution-generating technique is defined and a general class of solutions is obtained in which the polarization of the approaching waves is constant and aligned.

1. INTRODUCTION

The first exact solution describing the collision and interaction of plane electromagnetic waves was given by Bell and Szekeres (1974). Since then, a new approach to the topic has been developed by Chandrasekhar and Xanthopoulos (1985, 1987). This is based on the fact that the main field equations are identical to the Ernst equations for stationary axially symmetric space-times.

The structure of colliding plane wave solutions is illustrated in Figure 1. Two waves are assumed to approach each other in a flat background. The basic problem is to find the exact solution of the field equations in region IV for any given initial waves in regions II and III. In practice, however, it is easier first to find a solution in region IV, and then to consider the initial waves that give rise to it provided the appropriate boundary conditions are satisfied.

2. FIELD EQUATIONS

The line element for the space-time may conveniently be taken in the form

$$ds^2 = 2e^{-M}\, du\, dv - e^{-U}[\chi\, dy^2 + \chi^{-1}(dx - \omega\, dy)^2] \tag{1}$$

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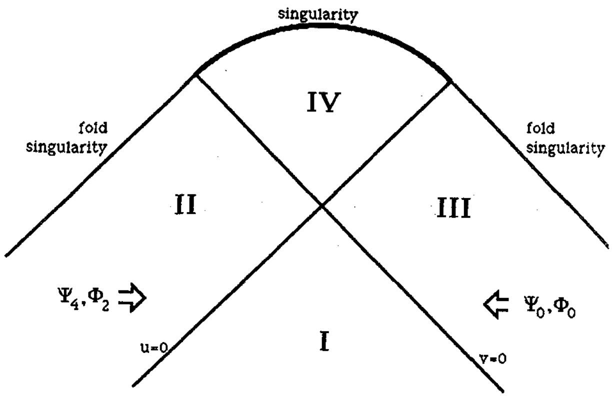

Fig. 1. The structure of colliding plane wave solutions. Region I is taken to be flat, regions II and III contain the approaching plane waves, and region IV is the interaction region. A strong curvature singularity usually develops in the interaction region, but this may be replaced by an unstable quasiregular singularity.

where U, χ, ω, and M are functions of the two null coordinates u and v only in the interaction region, are functions of u only in region II, are functions of v only in region III, and are constants in region I. The other coordinates x and y are aligned with the two spacelike Killing vectors. Using this notation, one can write the Einstein-Maxwell field equations in the form

$$U_{uv} = U_u U_v \tag{2}$$

$$\begin{aligned}2(\chi_{uv} + i\omega_{uv}) = {} & U_u(\chi_v + i\omega_v) + U_v(\chi_u + i\omega_u) + 2\chi^{-1}(\chi_u + i\omega_u)(\chi_v + i\omega_v) \\ & -4\chi(\chi^2 + \omega^2)^{-1/2}(\chi + i\omega)\Phi_0\bar{\Phi}_2\end{aligned} \tag{3}$$

$$2U_{vv} = U_v^2 - 2U_v M_v + \chi^{-2}(\chi_v^2 + \omega_v^2) + 4\Phi_0\bar{\Phi}_0 \tag{4}$$

$$2U_{uu} = U_u^2 - 2U_u M_u + \chi^{-2}(\chi_u^2 + \omega_u^2) + 4\Phi_2\bar{\Phi}_2 \tag{5}$$

$$2M_{uv} = -U_u U_v + \chi^{-2}(\chi_u \chi_v + \omega_u \omega_v) \tag{6}$$

$$\Phi_{2,v} = \frac{1}{2}\left(U_v - i\frac{\omega(\chi\chi_v + \omega\omega_v)}{\chi(\chi^2 + \omega^2)}\right)\Phi_2 + \frac{(\chi + i\omega)(\chi_u - i\omega_u)}{2\chi(\chi^2 + \omega^2)^{1/2}}\Phi_0 \tag{7}$$

$$\Phi_{0,u} = \frac{1}{2}\left(U_u + i\frac{\omega(\chi\chi_u + \omega\omega_u)}{\chi(\chi^2 + \omega^2)}\right)\Phi_2 + \frac{(\chi - i\omega)(\chi_v + i\omega_v)}{2\chi(\chi^2 + \omega^2)^{1/2}}\Phi_0 \tag{8}$$

Equations (7) and (8) imply that there exists a complex potential function $H(u, v)$ for the scale-invariant Maxwell components such that

$$\Phi_0 = -e^{U/2}\left[\frac{\chi(\chi - i\omega)}{(\chi^2 + \omega^2)^{1/2}}\right]^{1/2} H_v, \qquad \Phi_2 = e^{U/2}\left[\frac{\chi(\chi + i\omega)}{(\chi^2 + \omega^2)^{1/2}}\right]^{1/2} H_u \tag{9}$$

and H satisfies the equation

$$2\chi H_{uv}+(\chi_v+i\omega_v)H_u+(\chi_u-i\omega_u)H_v=0 \tag{10}$$

It can be seen that equation (2) can immediately be integrated to give

$$e^{-U}=f(u)+g(v) \tag{11}$$

where $f(u)$ and $g(v)$ are arbitrary functions that, in region I, must become constants that may be taken to be $\frac{1}{2}$.

Equations (4) and (5) imply that f and g are monotonically decreasing functions in the interaction region. In this region they may therefore be chosen as null coordinates instead of u and v. It may also be observed that the metric must become singular in the interaction region as $f+g\to 0$.

Finally, it can also be seen that the complex equation (3) is the integrability condition for the remaining equations. Thus, if a solution of (3) is found for χ and ω, then a function M automatically exists satisfying (4)-(6). Attention is thus focused on the two main equations (3) and (10).

It has been found convenient also to consider the alternative set of coordinates to f and g defined by

$$\begin{aligned} t&=(\tfrac{1}{2}-f)^{1/2}(\tfrac{1}{2}+g)^{1/2}+(\tfrac{1}{2}-g)^{1/2}(\tfrac{1}{2}+f)^{1/2}\\ z&=(\tfrac{1}{2}-f)^{1/2}(\tfrac{1}{2}+g)^{1/2}-(\tfrac{1}{2}-g)^{1/2}(\tfrac{1}{2}+f)^{1/2}\end{aligned} \tag{12}$$

With this,

$$e^{-U}=f+g=(1-t^2)^{1/2}(1-z^2)^{1/2} \tag{13}$$

and the singularity caused by the mutual focusing of the two waves occurs on the surface $t=1$.

At this point, it is convenient to adopt the approach of Chandrasekhar and Xanthopoulos (1985) and to introduce the function Ψ defined by

$$\Psi=(1-t^2)^{1/2}(1-z^2)^{1/2}\chi^{-1} \tag{14}$$

The imaginary part of (3) then implies that there exists a real potential function Φ such that

$$\begin{aligned}\omega_t&=\frac{1-z^2}{\Psi^2}[\Phi_z-i(H\bar{H}_z-\bar{H}H_z)]\\ \omega_z&=\frac{1-t^2}{\Psi^2}[\Phi_t-i(H\bar{H}_t-\bar{H}H_t)]\end{aligned} \tag{15}$$

It is then convenient to introduce the complex potential Z defined by

$$Z=\Psi+i\Phi+H\bar{H} \tag{16}$$

With this, the equations (3) and (10) can be written as the two complex Ernst equations

$$\begin{aligned}(\operatorname{Re} Z - H\bar{H})\nabla^2 Z &= (\nabla Z)^2 - 2\bar{H}(\nabla H)\cdot(\nabla Z)\\ (\operatorname{Re} Z - H\bar{H})\nabla^2 H &= (\nabla Z)\cdot(\nabla H) - 2\bar{H}(\nabla H)^2\end{aligned} \tag{17}$$

It is convenient to introduce the associated Ernst potentials E and η defined by

$$E = \frac{Z-1}{Z+1}, \qquad \eta = \frac{2H}{Z+1} \tag{18}$$

which satisfy the alternative equations

$$\begin{aligned}(1 - E\bar{E} - \eta\bar{\eta})\nabla^2 E &= -2\nabla E(\bar{E}\nabla E + \bar{\eta}\nabla\eta)\\ (1 - E\bar{E} - \eta\bar{\eta})\nabla^2 \eta &= -2\nabla\eta(\bar{E}\nabla E + \bar{\eta}\nabla\eta)\end{aligned} \tag{19}$$

3. BOUNDARY CONDITIONS

The appropriate junction conditions that should be used for colliding plane wave problems are those of O'Brien and Synge (1952). In terms of the metric functions of (1), these conditions require that U must be smooth, and that χ, ω, and M must be continuous across the null boundaries, which may be taken to be $u = 0$ and $v = 0$. It follows immediately that the functions $f(u)$ and $g(v)$ must have the form

$$\begin{aligned}&f = \tfrac{1}{2} \quad \text{for } u < 0, \qquad f = \tfrac{1}{2} - (c_1 u)^{n_1} + \cdots \quad \text{for } u \geq 0\\ &g = \tfrac{1}{2} \quad \text{for } v < 0, \qquad g = \tfrac{1}{2} - (c_2 v)^{n_2} + \cdots \quad \text{for } v \geq 0\end{aligned} \tag{20}$$

where $n_1, n_2 \geq 2$, and c_1 and c_2 are arbitrary constants related to the amplitudes of the approaching waves.

Any solution of the main field equations (17) or (19) will yield χ and ω as functions of f and g and, in view of (20), these will automatically satisfy the required boundary conditions. The difficulty occurs when the remaining field equations are integrated to obtain M.

It is now convenient to put

$$e^{-M} = \frac{f'g'}{(f+g)^{1/2}} e^{-S} \tag{21}$$

and equations (4) and (5) then imply that S satisfies

$$\begin{aligned}S_f &= -\tfrac{1}{2}(f+g)\left(\frac{\chi_f^2 + \omega_f^2}{\chi^2} + 4\frac{\Phi_2\bar{\Phi}_2}{f'^2}\right)\\ S_g &= -\tfrac{1}{2}(f+g)\left(\frac{\chi_g^2 + \omega_g^2}{\chi^2} + 4\frac{\Phi_0\bar{\Phi}_0}{g'^2}\right)\end{aligned} \tag{22}$$

Now M will only be continuous across $u=0$ and $v=0$ if S contains terms of the form

$$S=k_1\log(\tfrac{1}{2}-f)+k_2\log(\tfrac{1}{2}-g)+\cdots \tag{23}$$

where k_1 and k_2 are constants by

$$k_1=1-1/n_1,\qquad k_2=1-1/n_2 \tag{24}$$

and which therefore satisfy the inequality $\frac{1}{2}\leq k_1, k_2<1$.

The above conditions can alternatively be expressed in the form

$$\begin{aligned}\lim_{f\to 1/2}\left[(\tfrac{1}{2}-f)\left(\frac{\chi_f^2+\omega_f^2}{\chi^2}+\frac{4\chi}{f+g}H_f\bar{H}_f\right)\right]&=2\left(1-\frac{1}{n_1}\right)\\ \lim_{g\to 1/2}\left[(\tfrac{1}{2}-g)\left(\frac{\chi_g^2+\omega_g^2}{\chi^2}+\frac{4\chi}{f+g}H_g\bar{H}_g\right)\right]&=2\left(1-\frac{1}{n_2}\right)\end{aligned} \tag{25}$$

They may also be expressed in terms of the Ernst potentials in either of the forms

$$\begin{aligned}\lim_{\substack{t\to 0\\ z\to 0}}\left[\frac{Z_p\bar{Z}_p-2(\bar{H}H_p\bar{Z}_p+H\bar{H}_pZ_p)+2(Z+\bar{Z})H_p\bar{H}_p}{(Z+\bar{Z}-2H\bar{H})^2}\right]&=2\left(1-\frac{1}{n_1}\right)\\ \lim_{\substack{t\to 0\\ z\to 0}}\left[\frac{Z_q\bar{Z}_q-2(\bar{H}H_q\bar{Z}_q+H\bar{H}_qZ_q)+2(Z+\bar{Z})H_q\bar{H}_q}{(Z+\bar{Z}-2H\bar{H})^2}\right]&=2\left(1-\frac{1}{n_2}\right)\end{aligned} \tag{26}$$

or

$$\begin{aligned}\lim_{\substack{t\to 0\\ z\to 0}}\left[\frac{(1-\eta\bar{\eta})E_p\bar{E}_p+\bar{\eta}E\eta_p\bar{E}_p+\eta\bar{E}\bar{\eta}_pE_p+(1-E\bar{E})\eta_p\bar{\eta}_p}{(1-E\bar{E}-\eta\bar{\eta})^2}\right]&=2\left(1-\frac{1}{n_1}\right)\\ \lim_{\substack{t\to 0\\ z\to 0}}\left[\frac{(1-\eta\bar{\eta})E_q\bar{E}_q-\bar{\eta}E\eta_q\bar{E}_q-\eta\bar{E}\bar{\eta}_qE_q+(1-E\bar{E})\eta_q\bar{\eta}_q}{(1-E\bar{E}-\eta\bar{\eta})^2}\right]&=2\left(1-\frac{1}{n_2}\right)\end{aligned} \tag{27}$$

where, for convenience, I have put

$$\frac{\partial}{\partial p}=\frac{\partial}{\partial t}+\frac{\partial}{\partial z},\qquad \frac{\partial}{\partial q}=\frac{\partial}{\partial t}-\frac{\partial}{\partial z} \tag{28}$$

4. SOLUTION-GENERATING TECHNIQUES

Many techniques are known by which electrovac solutions of the Ernst equations can be generated from initial solutions of the vacuum equations, particularly in situations such as this in which there are two commuting Killing vectors. In particular, it can be shown that the Ehlers–Harrison transformation will always generate a new colliding wave solution if the initial vacuum solution describes colliding plane gravitational waves (Li and Ernst, 1989). Some such solutions have been obtained by García Díaz (1988, 1989) [see also the general review of Griffiths (1990)].

For example, it can easily be seen that, if $E=E_0$, $\eta=0$ is a vacuum solution of (19) satisfying (27), then a family of new electrovac solutions which automatically satisfies (27) is given by

$$E=aE_0+b, \qquad \eta=cE_0+d \tag{29}$$

where a, b, c, and d are arbitrary complex constants that are only required to satisfy the two conditions

$$a\bar{b}+c\bar{d}=0, \qquad a\bar{a}+b\bar{b}+c\bar{c}+d\bar{d}=1 \tag{30}$$

Chandrasekhar and Xanthopoulos (1985, 1987) have obtained a number of solutions using this transformation both with $a=0$, $b=0$, $c=1$, and $d=0$, and also with $b=0$, $c=0$, and $d=(1-a^2)^{1/2}$. The above transformation with $b=0$, $d=0$, and $c=(1-a^2)^{1/2}$ has also been used by Halilsoy (1988).

It may be noted, however, that the solutions obtained in this way are generalizations of colliding gravitational wave solutions and that, provided $a\neq 0$, they revert to these solutions in the limit as the electromagnetic components vanish. It is therefore expected that any exact solutions obtained will describe the collision of electromagnetic waves that are coupled with gravitational waves (even in the case when $a=0$). The condition that the approaching waves are purely electromagnetic, and that regions II and III are conformally flat, can be simply stated as the condition that $\chi\to 1$ and $\omega\to 0$ on both of the boundaries as $f\to\frac{1}{2}$ and as $g\to\frac{1}{2}$. However, it is not clear how this can be expressed as general conditions on the Ernst potentials, although this condition may be satisfied in some particular cases such as in the Bell-Szekeres solution.

5. DIAGONAL SOLUTIONS

The class of solutions in which the metric can be diagonalized is characterized by the fact that $\omega=0$. This implies that $\Phi_0\bar{\Phi}_2$ must be real.

I now restrict attention to the case when the Ernst potentials Z and H, or E and η, and Φ_0 and Φ_2 are all real. In this case, the approaching electromagnetic waves have constant aligned polarization and, if these waves are also coupled with gravitational waves, then the polarization vectors of the approaching gravitational components must also be aligned. This condition can be expressed by the fact that Ψ_0 and Ψ_4 must both be real and must give rise to a real component Ψ_2 in the interaction region.

A class of exact solutions of this type can now be obtained using as an initial solution a general vacuum solution $E=E_0$, $\eta=0$ of (18), where E_0 is real. It is convenient to put

$$\frac{1+E_0}{1-E_0}=e^{-U_0}\chi_0^{-1}=(1-t^2)^{1/2}(1-z^2)^{1/2}\,e^{V_0} \tag{31}$$

so that the main vacuum field equation becomes the single real linear equation

$$[(1-t^2)V_{0,t}]_{,t}-[(1-z^2)V_{0,z}]_{,z}=0 \tag{32}$$

A large class of solutions of this equation can be written in the form

$$V_0=\sum_n [a_nP_n(t)P_n(z)+q_nQ_n(t)P_n(z)+p_nP_n(t)Q_n(z)+b_nQ_n(t)Q_n(z)]$$
$$-\tfrac{1}{2}k\log(1-t^2)-\tfrac{1}{2}k\log(1-z^2)+\sum_i d_i\cosh^{-1}\left(\frac{c_i-tz}{(1-t^2)^{1/2}(1-z^2)^{1/2}}\right) \tag{33}$$

where $P(x)$ and $Q(x)$ are Legendre functions of the first and second kinds, respectively, and a_n, p_n, q_n, b_n, c_i and d_i are sets of arbitrary constants. In order to satisfy the boundary conditions, V_0 must contain either at least one Legendre function of the second kind, or two $\cosh^{-1}$ terms with $c_1=-1$ and $c_2=1$.

A general family of colliding electromagnetic wave solutions can now be obtained using the transformation (29) with the constants a, b, c, and d real and satisfying (30), and with the initial solution E_0 given by (31) and (33). Following Chandrasekhar and Xanthopoulos (1987), it is convenient to put

$$F=\tfrac{1}{2}(V_0-U_0) \tag{34}$$

so that

$$E_0=\tanh F \tag{35}$$

With this, the complete solution can now be expressed in the form

$$\begin{aligned} &U=U_0, && \chi=\frac{1}{a^2+c^2}\,e^{-U_0}[(1-b)\cosh F-a\sinh F]^2\\ &\omega=0, && e^{-M}=e^{2F}[(1-b)\cosh F-a\sinh F]^2\,e^{-M_0}\\ &\Phi_0=-\frac{(c-bc+ad)F_v}{(1-b)\cosh F-a\sinh F}, && \Phi_2=\frac{(c-bc+ad)F_v}{(1-b)\cosh F-a\sinh F}\end{aligned} \tag{36}$$

It is of interest to note that, in the case when $b=0$, $d=0$, and $c=(1-a^2)^{1/2}$, the above technique is equivalent to that obtained previously by Panov (1978, 1979), and used by him to obtain a general class of solutions with a strong curvature singularity in the interaction region.

In the further particular case when $a=0$, $b=0$, $c=1$, and $d=0$, the above technique was given by Chandrasekhar and Xanthopoulos (1987). [It is of interest to note that the solutions obtained in this way had previously been obtained by Charach (1979), who described them in terms of electromagnetic Gowdy cosmologies, without applying the boundary conditions for colliding plane waves.] It may also be noted that the Bell–Szekeres solution is included in this particular case when

$$F=-P_0(z)Q_0(t) \tag{37}$$

(The initial vacuum solution in this case is the degenerate Ferrari–Ibañez solution, which is part of the Schwarzschild solution inside the initial horizon.) Chandrasekhar and Xanthopoulos (1987) also obtained a class of solutions which similarly do not contain strong curvature singularities on the surface $t=1$ by putting

$$F=\tfrac{1}{2}\sum_{n=0}^{\infty} a_n P_n(t)P_n(z)-P_0(z)Q_0(t) \tag{38}$$

in this particular case.

A much larger class of solutions that does not contain a strong curvature singularity when $t=1$ can be obtained by using the general transformation (36) and by taking initially the vacuum solution of Feinstein and Ibañez (1988). In this case

$$\begin{aligned} F=&\tfrac{1}{2}(k-1)U_0+\tfrac{1}{2}\sum_n a_n P_n(t)P_n(z)\\ &+\tfrac{1}{2}\sum_i d_i \cosh^{-1}\left(\frac{c_i-tz}{(1-t^2)^{1/2}(1-z^2)^{1/2}}\right) \end{aligned} \tag{39}$$

where

$$\sum d_i = k\pm 1 \tag{40}$$

The necessary boundary conditions are satisfied in this case provided there are at least two $\cosh^{-1}$ (solitonic) terms with constants satisfying

$$c_1=-1, \qquad c_2=1, \qquad d_1^2=2(1-1/n_1), \qquad d_2^2=2(1-1/n_2) \tag{41}$$

where $n_1, n_2 \geq 2$. In these solutions it is possible for the wavefronts to be continuous. It is also of interest to note that the two solitonic terms that provide the continuity across the boundaries of the interaction region in this case can be expressed in the alternative forms

$$\begin{aligned} &d_1\cosh^{-1}\left(\frac{1+tz}{(1-t^2)^{1/2}(1-z^2)^{1/2}}\right)+d_2\cosh^{-1}\left(\frac{1-tz}{(1-t^2)^{1/2}(1-z^2)^{1/2}}\right)\\ &\quad=-(d_1+d_2)Q_0(t)P_0(z)-(d_1-d_2)P_0(t)Q_0(z) \end{aligned} \tag{42}$$

In all these solutions the surface $t = 1$ corresponds to an unstable quasi-regular singularity.

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